LOWER BOUND OF ZERO-VISIBILITY COPS AND ROBBER GAME ON THE CUBE GRID

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ABSTRACT

Zero-visibility cops and robber game is a variant of the classical model. The robber is invisible in the zero-visibility game. In this paper, we study the zero-visibility cops and robber game on the cube grid. We first study a partition problem of the cube grid C_{n^3}. Then we prove the lower bound on the zero-visibility cop number of the cube grid by using the results in the partition. We also show the lower bound is a quadratic polynomial about n.

KEYWORDS

Cube Grid; Cops and robber; Partition; Zero-visibility Cop number

1. INTRODUCTION

Cops and robber game is an interesting pursuit-evasion game in graph theory, which was introduced by Nowakowski and Winkler [1] and Quilliot [2] in 1983. We consider the game of cops and robber on a graph G. There are two kinds of players: one robber and multiple cops. In each round, the cops move first and then the robber. The player can only move to the adjacent vertex or stay still. The robber is captured if a cop and the robber occupy the same vertex in a finite number of moves. Our goal is to determine the minimum number of cops to capture the robber.

Cops and robber game can be divided into three models based on visibility. In these three models, the robber knows the location of all cops at any time, but the cop's visibility of the robber is different. The first is that the cops know the perfect information of the robber, a lot of results can be found in [3,4,5,6]. The second is that the cops know the imperfect information of the robber. The visibility of the cops is limited to the distance between the cops and the robber within k range, so this model is called the k-visibility cops and robber game. In this case, related results can be found in [7,8].

Zero-visibility cops and robber game is the third model, in which the cops have no information about the location of the robber, the robber is invisible. The zero-visibility cop number of G, denoted $c_0(G)$, is the minimum number of cops needed to capture the invisible robber, and a strategy used $c_0(G)$ cops is called the optimal search strategy. It was first proposed by Tošić [9] in 1985. Zero-visibility model makes the problem more difficult, so there are not many results. The key to determine $c_0(G)$ is to find the lower bound. Yang [10] proposed a partition method, which can be applied to prove the lower bound of $c_0(G)$. this method only used for simple

cartesian products in [10]. Cube grid is a three-dimensional graph, so we use this method on the cube grid. Given an integer k, and find a partition of the cube grid such that the size of the boundary of the smaller subset in the partition is at most k while the size of the subset is as large as possible [10]. This method can successfully help us determine the lower bound of $c_0(G_{n^3})$,

The rest of this paper is as follows. In section 2, we give the related work. In section 3, we give some definitions. In section 4, we study a partition problem of the cube grid. In section 5, we show the lower bound of $c_0(C_{n^3})$ is a quadratic polynomial about n, and the last section is a summary.

2. RELATED WORK

Tošić [9] studied the zero-visibility cop numbers of paths, cycles, complete graphs, and complete bipartite graphs. Tang [11] proposed a quadratic time algorithm to determine the zero-visibility cop number of a tree, then a linear algorithm of a tree was given by Dereniowskl [12]. In the above articles, only simple graphs were studied, and there is no better method to determine the zero-visibility cop number until Dereniowskl [13] studied the relationship between the monotonic zero-visibility cop number and the path width of a graph. Some scholars have calculated the path width of different graphs in [14], also the path width of the cube grid can be found in [15]. But how to determine the lower bound of $c_0(G)$ is also difficult. Yang [10] proposed to use the partition method to prove the lower bound of the cop number in a simple Cartesian product graph, and new results of this method to determine the zero-visibility cop number can be found in [16].

3. PRELIMINARIES

Consider the zero-visibility cops and robber game on a graph $G(V,E)$ with vertex set V and edge set E, which can be seen as a node-search problem. First, make all vertices contaminated, then the robber can only be on contaminated vertices. Thus, the search is successful if the givencop number and search strategy can make all vertices clean.

Let G and G' be two graphs, the cartesian product of G and G', denoted by $G \square G'$, is the graph in which $V(G \square G') = V(G) \times V(G')$, and two vertices (u,v) and (u',v') are adjacent in $G \square G'$ if and only if $u = u'$, v and v' are adjacent in G'; or $v = v'$, u and u' are adjacent in G. Let P_n be a path with n vertices. The square grid is the cartesian product of P_n and P_n, denoted by $C_{n^2}(n \geq 2)$. The cube grid is the cartesian product of C_{n^2} and P_n, denoted by $C_{n^3}(n \geq 2)$, which has n^3 vertices. we define the direction in the cubic grid, let $d:\{x,y,z\}$, see in Figure 1. In each direction, there are n copies of C_{n^2} and n^2 copies of P_n. C_{3^2} and C_{3^3} shown in Figure 1.

Let (V_1,V_2) be a partition of $G(V,E)$ if $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$. Two vertices v_i and v_j are adjacent, denoted by $v_i \sim v_j$. Let V_B be the boundary of V_1 if $V_B \subseteq V_1$, $V_B = \{v_i \mid v_i \in V_1, \exists v_j \in V_2 \ and \ v_i \sim v_j\}$, $|V_B|$ to be largest. The vertex in V_1 not in V_B, denoted by

V_C, we have $V_C = V_1 - V_B$. In the zero-visibility cops and robber searching model, the contaminated and cleared vertices always changed after each round. Let $V_B^i (i \geq 1)$ be the boundary after the cops' move in round i, V_1^i, V_2^i be the set of the cleared and contaminated vertices after the robber' move in round i. (V_1^i, V_2^i) is a partition of G after the robber' move in round i.

Each direction contains all vertices of the cube grid, so we take the z direction as an example. Let $C_{n^2}^l (1 \leq l \leq n)$ be the l-th C_{n^2} in the z direction, $v_{l,t} (1 \leq t \leq n^2)$ be the t-th vertex in the $C_{n^2}^l$ (see in figure 1). Let r_l, p_l, q_l be the number of vertices in V_1, V_C, V_B of $C_{n^2}^l (1 \leq l \leq n)$ respectively, we have $r_l = p_l + q_l$.
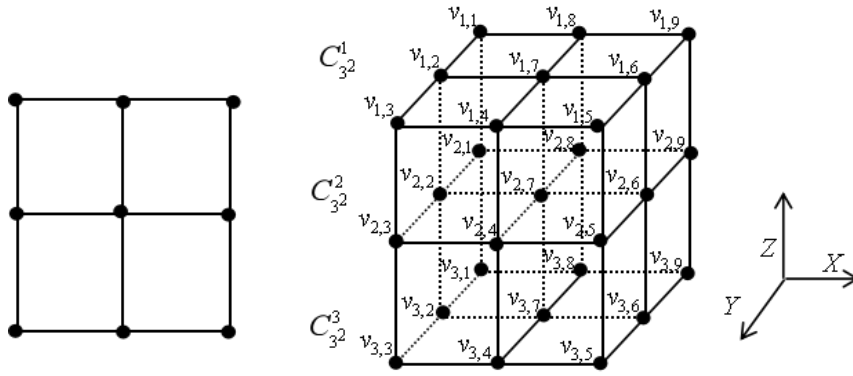

Figure 1. Square grid C_{3^2}, Cube grid C_{3^3}.

4. PARTITION OF THE CUBE GRID

Lemma 1. Let (V_1, V_2) be a partition of C_{n^2} for $n \geq 2$. If $|V_1| = \dfrac{m(m+1)}{2}$ and $1 \leq m \leq n$, then $|V_B| \geq m$.

Proof. The claim is true when $m = 1$. For the sake of contradiction, suppose that $|V_B| < m$ when $2 \leq m \leq n$. If there exist m rows or m columns that contain both vertices in V_1 and V_2, then $|V_B| \geq m$, which is a contradiction. Hence, there are at least $n - m + 1$ rows and $n - m + 1$ columns that contain vertices only in V_1 or V_2. Here we have two cases.

Case1: There exist $n - m + 1$ rows and $n - m + 1$ columns that contain vertices only in V_2. In this case, it can be seen that there exist a row and a column contain vertices only in V_2 of C_{m^2}. From theorem 1 in Yang [15], we have $|V_1| \leq \dfrac{m(m-1)}{2} < \dfrac{m(m+1)}{2}$.

Case2: There exist $n-m+1$ rows and $n-m+1$ columns that contain vertices only in V_1. Note that $|V_1|_{\min} = n^2 - |V_2|_{\max}$, the maximum of $|V_2|$ is equal to the maximum of $|V_C|$ in case 1, so consider case 1, we will show that when $|V_C|_{\max} = w$, $|V_B| = m-1$. Assume that $|V_B| = k < m-1$, $|V_C|_{\max} = w$. If there are $m-1$ rows or $m-1$ columns contain both vertices in V_1 and in V_2, then $|V_B| \geq m-1$, which is a contradiction. Thus, there are at least $n-m+2$ rows and $n-m+2$ columns that contain vertices only in V_2. Let v_i be a vertex in V_2, satisfied $\{v_i \mid v_i \sim v_j, v_j \in V_B\}$, then let $v_i \in V_1$, we can obtain a new partition of C_{n^2}. In a new partition, we have $|V_B| = k$, $|V_C| = w+1$; or $|V_B| = k+1$, $|V_C| = w$, which is a contradiction. Thus, when $|V_C|_{\max} = w$, we obtain $|V_B| = m-1$. Since $w = |V_1| - (m-1)$, $|V_1| \leq \dfrac{m(m-1)}{2}$, we have $w \leq \dfrac{m(m-1)}{2} - (m-1)$.

$$|V_1|_{\min} = n^2 - [\frac{m(m-1)}{2} - (m-1)]$$

Hence, in case 2, . Therefore, $|V_1| \geq \dfrac{m(m+3)}{2} - 1 > \dfrac{m(m+1)}{2} (m \geq 2)$.
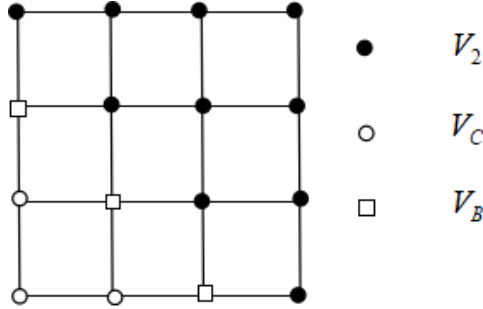
See in figure 2.

Figure 2. C_{4^2} with $m=3$, $|V_1|=6$, $|V_B|=3$.

Lemma 2. Let (V_1, V_2) be a partition of C_{n^3} for $n \geq 2$. If $\max\{r_l \mid 1 \leq l \leq n\} = m(m \leq \dfrac{n(n+1)}{2})$, then $|V_B| \geq m$.

Proof. We know that on a $C_{n^3} (n \geq 2)$, if $r_{l_1} \leq r_{l_2} \leq \dfrac{n(n+1)}{2}, (1 \leq l_1, l_2 \leq n)$, then $q_{l_1} \leq q_{l_2}$. So $|V_B| = \sum\limits_{l=1}^{n} q_l$ is the minimum when $r_1 = m$ or $r_n = m$, this two cases are similar. Thus, take $r_n = m$ as an example, there will be $r_{l-1} \leq r_l$, $q_{l-1} \leq q_l (2 \leq l \leq n)$. Here are two cases.

Case1: $\min\{r_l \mid 1 \le l < n\} = 0$ .Suppose that $r_i = 0$, then let $v_{m,t}$ be a vertex in $C_{n^2}^m$ such that $v_{m,t} \in V_1$, there exist a path $v_{i,t}...v_{m,t}$ $(v_{i,t} \in C_{n^2}^i, v_{i,t} \in V_2)$ in z direction, and this path must contain a vertex in V_B. Since $r_i = m$, we have $|V_B| \ge m$. (see in figure 3)

Case2: $\min\{r_l \mid 1 \le l < n\} > 0$ .Suppose that $r_n = \dfrac{n(n+1)}{2}$, then $q_n \ge n$ from lemma 1. And let $q_n = n$ for minimum, we obtain $1 \le q_l \le n$, and we have $q_{l-1} \le q_l$ when $|V_B|$ is the minimum, so there exists one copy of C_{n^2} that contain one vertex in V_B. Since $r_n = m < \dfrac{n(n+1)}{2}$, there are at least two copies of C_{n^2} in which the number of boundaries equals 1 when $|V_B|$ is the minimum. Let $C_{n^2}^j$ be a copy of C_{n^2} in which $r_j = q_j = 1$. Suppose that there is no copy of C_{n^2} between $C_{n^2}^j$ and $C_{n^2}^n$ in which the boundary number equals 1, similar to (1), we have $|V_B| \ge m$. And there exist another copy of C_{n^2} whose boundary number equals 1, so we have $|V_B| \ge m+1$.(see in figure 4)
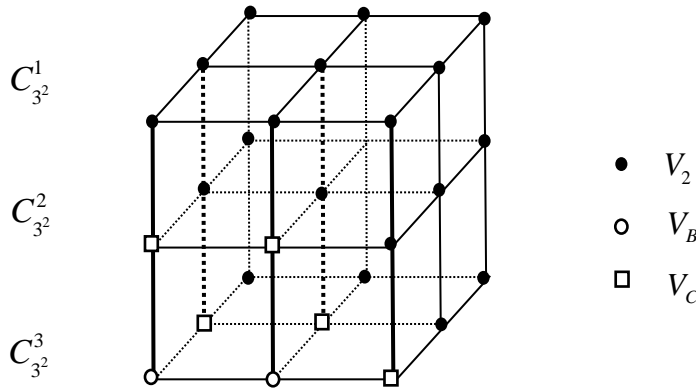

Figure 3. C_{3^3} with $i = 1, r_i = 0, m = 5$, and there exist 5 copies of P_3 (bold lines)which contain a vertex in V_1 and a vertex in V_2, so $|V_B| = 5$.

Theorem 1. Let (V_1, V_2) be a partition of C_{n^3} for $n \ge 2$. If $|V_B| < \dfrac{n(n+1)}{2} + 1$, and there exist $\dfrac{n(n-1)}{2}$ copies of P_n in which all vertices belong to V_2 in one direction, then $|V_1| \le \sum_{l=1}^{n} \dfrac{l(l+1)}{2}$.

Proof. Take z direction as an example. Since there exist $\dfrac{n(n-1)}{2}$ copies of P_n whose vertices all belong to V_2, then $r_l \le \dfrac{n(n+1)}{2} (1 \le l \le n)$. We need to find the maximum of $|V_1|$. Suppose that $\max\{r_l(1 \le l \le n)\} = \dfrac{n(n+1)}{2}$, from lemma2, we have $|V_B| \ge \dfrac{n(n+1)}{2}$.And since

$|V_B| < \dfrac{n(n+1)}{2} + 1$, we have $|V_B| = \dfrac{n(n+1)}{2}$. If $C_{n^2}^i, C_{n^2}^{i+1} \ldots \ldots C_{n^2}^j (1 \le i \le j \le n)$ contain vertices in V_1, then $r_{k-1} = p_{k-1} + q_{k-1} \ge p_k (i+1 \le k \le j)$. Hence, when $|V_B| = \dfrac{n(n+1)}{2}$, there will be

$r_{k-1} = p_k (i+1 \le k \le n)$, $p_i = 0$, $r_i = q_i$, $r_n = \dfrac{n(n+1)}{2}$. And from lemma1, $q_n \ge n$. Since $r_{k-1} = p_k$, we have $r_{n-1} = p_n \le \dfrac{n(n+1)}{2} - n = \dfrac{n(n-1)}{2}$. Let $r_{n-1} = \dfrac{n(n-1)}{2}$ for maximum, also from lemma 1, there will be $q_{n-1} \ge n-1$, and so on. Thus, when $|V_1|$ is the maximum, we have $r_l = \dfrac{l(l+1)}{2}$,

$q_l = l(1 \le l \le n)$, satisfied $|V_B| = \sum\limits_{l=1}^{n} q_l = \dfrac{n(n+1)}{2} < \dfrac{n(n+1)}{2} + 1$. Therefore,

$|V_1| \le \sum\limits_{l=1}^{n} \dfrac{l(l+1)}{2}$. (see in figure 4)
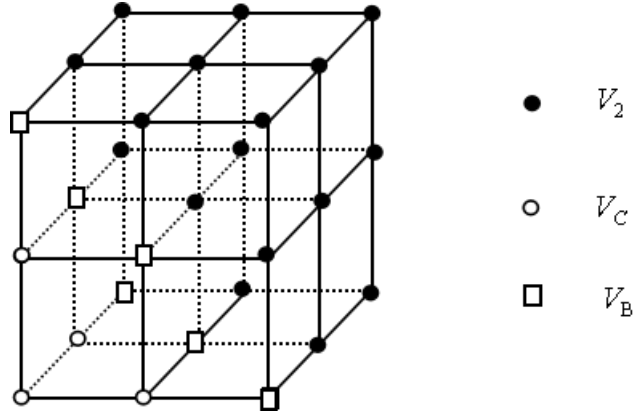

Figure 4. C_{3^3} with $|V_B| = 6$, $|V_1| = 10$.

5. LOWER BOUND OF ZERO-VISIBILITY COP NUMBER

Lemma 3([15]). Let G be a graph with $c_0(G) = k$. For any round $i(i \ge 1)$ of an optimal search strategy of G, if $|V_B^i| \ge 2k$, then $|V_1^i| \le |V_1^{i-1}|$.

Lemma4. For an optimal search strategy of C_{n^3}, if $|V_B^i| < \dfrac{n(n+1)}{2} + 1$, then there will be $|V_1^i| \le \sum\limits_{l=1}^{n} \dfrac{l(l+1)}{2}$ or $|V_1^i| \ge n^3 - \sum\limits_{l=1}^{n} \dfrac{l(l+1)}{2} + \dfrac{n(n+1)}{2}$.

Proof. After cops' move in round i, take z direction as an example, there are n^2 copies of P_n in z direction. If there are $\dfrac{n(n+1)}{2}+1$ copies of P_n that contain both cleared vertices and contaminated vertices, then $|V_B^i| \geq \dfrac{n(n+1)}{2}+1$, which is a contradiction. Hence, there are at least $\dfrac{n(n-1)}{2}$ copies of P_n in which all vertices are cleared or contaminated. Here we have two cases.

Case1: There exist $\dfrac{n(n-1)}{2}$ copies of P_n that contain only contaminated vertices. Let V_1, V_2 be the set of cleared vertices and contaminated vertices respectively after cops' move in round i. From theorem1, we have $|V_1| \leq \sum\limits_{l=1}^{n}\dfrac{l(l+1)}{2}$. Therefore, $|V_1^i| \leq |V_1| \leq \sum\limits_{l=1}^{n}\dfrac{l(l+1)}{2}$.

Case2: There exist $\dfrac{n(n-1)}{2}$ copies of P_n that contain only cleared vertices. The proof is similar to lemma1, consider case 1 again, we will show that when $|V_B^i| = \dfrac{n(n+1)}{2}$, $|V_1|-|V_B^i|$ is the maximum. Thus, in case2, the maximum number of contaminated vertices is $\sum\limits_{l=1}^{n}\dfrac{l(l+1)}{2}-\dfrac{n(n+1)}{2}$. Therefore, we have $|V_1^i| \geq n^3 - \sum\limits_{l=1}^{n}\dfrac{l(l+1)}{2}+\dfrac{n(n+1)}{2}$.

Theorem 2. For $n \geq 2$, $c_0(C_{n^3}) \geq \left\lfloor \dfrac{n(n+1)}{4}+\dfrac{3}{2} \right\rfloor$.

Proof. For the sake of contradiction, suppose that $c_0(C_{n^3}) \leq \left\lfloor \dfrac{n(n+1)}{4}+\dfrac{1}{2} \right\rfloor$, and consider a strategy that uses at most $\left\lfloor \dfrac{n(n+1)}{4}+\dfrac{1}{2} \right\rfloor$ cops, we will show that $|V_1^i| \leq \sum\limits_{i=1}^{n}\dfrac{l(l+1)}{2} < n^3, (i \geq 0)$.

That means $\left\lfloor \dfrac{n(n+1)}{4}+\dfrac{1}{2} \right\rfloor$ cops can't make all vertices clean, which is a contradiction. When $i=0$, we have $|V_1^0| \leq \left\lfloor \dfrac{n(n+1)}{4}+\dfrac{1}{2} \right\rfloor \leq \sum\limits_{l=1}^{n}\dfrac{l(l+1)}{2}$; then assume that $|V_1^{i-1}| \leq \sum\limits_{l=1}^{n}\dfrac{l(l+1)}{2}$, here we have two cases to prove $|V_1^i| \leq \sum\limits_{l=1}^{n}\dfrac{l(l+1)}{2}$.

Case1: $|V_B^i| \geq \dfrac{n(n+1)}{2}+1$. From lemma 3, $|V_1^i| \leq |V_1^{i-1}|$. Therefore, $|V_1^i| \leq \sum\limits_{l=1}^{n}\dfrac{l(l+1)}{2}$.

Case2: $|V_B^i| < \dfrac{n(n+1)}{2} + 1$. Since $|V_1^{i-1}| \le \sum\limits_{l=1}^{n} \dfrac{l(l+1)}{2}$, there are at most

$\left\lfloor \dfrac{n(n+1)}{4} + \dfrac{1}{2} \right\rfloor + \sum\limits_{l=1}^{n} \dfrac{l(l+1)}{2}$ cleared vertices after cops' move in round i . Also, since

$n^3 - \sum\limits_{l=1}^{n} \dfrac{l(l+1)}{2} + \dfrac{n(n+1)}{2} > \left\lfloor \dfrac{n(n+1)}{4} + \dfrac{1}{2} \right\rfloor + \sum\limits_{l=1}^{n} \dfrac{l(l+1)}{2}$, we have $|V_1^i| \le \sum\limits_{l=1}^{n} \dfrac{l(l+1)}{2}$ from

lemma 4.

6. CONCLUSION

In this paper, we successfully apply the partition method to study zero-visibility cops and robber game on the cube grid and show the lower bound of $c_0(C_{n^3})$. Perhaps the lower bound obtained by this method is not the best, But we narrow the scope of finding the final solution. In the future, we will use the result in this paper to find the exact value of $c_0(C_{n^3})$ and also use this method on other three-dimensional graph.

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