THE COLORED DISCONNECTION NUMBERS OF CELLULAR AND GRID NETWORKS

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ABSTRACT

Let G be a nontrivial link-colored connected network. A link-cut R of G is called a rainbow link-cut if no two of its links are colored the same. A link-colored network G is rainbow disconnected if for every two nodes u and v of G, there exists a u-v rainbow link-cut separating them. Such a link coloring is called a rainbow disconnection coloring of G. For a connected network G, the rainbow disconnection number of G, denoted by rd(G), is defined as the smallest number of colors that are needed in order to make G rainbow disconnected. Similarly, there are some other new concepts of network colorings, such as proper disconnection coloring, monochromatic disconnection coloring and rainbow node-disconnection coloring.

In this paper, we obtain the exact values of the rainbow (node-)disconnection numbers, proper and monochromatic disconnection numbers of cellular networks and grid networks, respectively.

KEYWORDS

link- (node-)coloring, connectivity, rainbow link- (node-)cut, (strong) rainbow (node-)disconnection numbers, proper and monochromatic disconnection numbers, cellular network, grid network

1. INTRODUCTION

All networks (also called graphs) considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a nontrivial connected network with node set $V(G)$ and link set $E(G)$. The *order* of G is denoted by $n = |V(G)|$. For a node $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) | uv \in E(G)\}$ and $d(v) = |N(v)|$ is the *degree* of v, and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Denote by P_n a path on n *nod*es. For a subset S of $V(G)$, we use $G[S]$ to denote the subnetwork of G *induced by S*. Let V_1, V_2 be two disjoint node subsets of G. We denote the set of links between V_1 and V_2 in G by $E(V_1, V_2)$. We follow [7] for network theoretical notation and terminology not defined here.

The concept of rainbow connection coloring was introduced by Chartrand et al. [10] in 2008. A rainbow path is a path whose links are colored pairwise differently. A link-coloring of a network G is a rainbow connection coloring if any two nodes of G are connected by a rainbow path. The rainbow connection number of a connected network G, denoted by rc(G), is the minimum number of colors so that G has a rainbow connection coloring. Rainbow node-connection was proposed by Krivelevich and Yuster [12] in 2010. For more details about the rainbow (node-)connection, we refer to [13] and survey papers [14, 16] and book [15].

As we know that there are two ways to study the connectivity of a network, one way is by using paths and the other is by using cuts. The above rainbow connection and rainbow node-connection use paths.

David C. Wyld et al. (Eds): SAIM, ACSIT, SNLP, ICITE, CoNeCo - 2021
pp. 101-112, 2021. CS & IT - CSCP 2021 DOI: 10.5121/csit.2021.111009

So, it is natural to consider the rainbow link-cuts and rainbow node-cuts for the colored connectivity in colored networks.

In [8], Chartrand et al. first studied the rainbow link-cuts by introducing the concept of rainbow disconnection of networks, and later produced some other new concepts of colored disconnection colorings, such as proper disconnection coloring and monochromatic disconnection coloring. Let G be a nontrivial connected and link-colored network. A *link-cut* of G is a set R of links of G such that $G - R$ is disconnected. If all (adjacent, no two) links in R have different colors, then R is called a *rainbow (proper, monochromatic) link-cut*. Let u and v be two nodes. A rainbow (proper, monochromatic) link-cut R is called a *u-v rainbow (proper, monochromatic) link-cut* if the nodes u and v belong to different components of $G–R$. A link-coloring of G is called a *rainbow (proper, monochromatic) disconnection coloring* (abbreviated as rd-*coloring,* pd-*coloring* and md-*coloring*) if for every two distinct nodes u and v of G, there exists a u-v rainbow (proper, monochromatic) link-cut in G, separating them. The *rainbow (proper) disconnection number* (abbreviated as rd (pd)-*number*) rd(G) (pd(G)) of G is the minimum number of colors required by a rainbow (proper) disconnection coloring of G. The *monochromatic disconnection number* (abbreviated as md-*number*) md(G) of G is the maximum number of colors required by a monochromatic disconnection coloring of G.

In fact, the rainbow disconnection number has the following application background. In some illegal commodity transactions, we hope to stop the transaction in time and send out a signal (a certain frequency). On the one hand, we need to block all the roads between the two cities and identify the interception locations based on different signals; on the other hand, we want to use as few frequencies as possible in order to reduce costs. Therefore, we want to know what is the minimum frequency required to meet the above requirements? Treat each city as a node. If there is a road between two cities, we add a link between the two nodes, and use G to denote the resulting network. Give a link-coloring for G, where the color on the link corresponds to the frequency of the road. Therefore, the above problem is equivalent to calculating the rainbow disconnection number of the network G.

In order to study the rainbow node-cut, we introduce the concept of rainbow node-disconnection number in this paper. For a connected and node-colored network G, let x and y be two nodes of G. If x and y are nonadjacent, then an *x-y node-cut* is a subset S of $V(G)$ such that x and y belong to different components of $G - S$. If x and y are adjacent, then an *x-y node-cut* is a subset S of $V(G)$ such that x and y belong to different components of $(G - xy) - S$. A node subset S of G is *rainbow* if no two nodes of S have the same color. An *x-y rainbow node-cut* is an x-y node-cut S such that if x and y are nonadjacent, then S is rainbow; if x and y are adjacent, then $S + x$ or $S + y$ is rainbow.

A node-colored network G is called *rainbow node-disconnected* if for any two nodes x and y of G, there exists an x-y rainbow node-cut. In this case, the node-coloring c is called a *rainbow node-disconnection coloring* of G. For a connected network G, the *rainbow node-disconnection number* of G, denoted by rnd(G), is the minimum number of colors that are needed to make G rainbow node-disconnected. A rainbow node-disconnection coloring with rnd(G) colors is called an rnd-*coloring* of G.

Remember that in the Menger's Theorem, only minimum link-cuts play a role, however, in the definition of rd-colorings we only requested the existence of a u-v link-cut between nodes u and v, which could be any link-cut (large or small are both OK). This may cause the failure of a colored version of such a nice Min- Max result. In order to overcome this problem, we introduced the concept of strong rainbow disconnection in networks in [5], with a hope to set up the colored version of the so-called Max-Flow Min-Cut Theorem.

A link-colored network G is called *strong rainbow disconnected* if for every two distinct nodes u and v of G, there exists a both rainbow and minimum u-v link-cut (*rainbow minimum u-v link-cut* for short) in G. Such a link-coloring is called a *strong rainbow disconnection coloring* (abbreviated as srd-

coloring) of G. For a connected network G, similarly, the *strong rainbow disconnection number* (abbreviated as srd-*number*) of G, denoted by srd(G), is the minimum number of colors required to make G strong rainbow disconnected. We call the colored (dis)connection numbers the global chromatic numbers, and the classic or traditional chromatic numbers the local chromatic numbers [6].

The rapid development of computer networks and communication technology, and the rise and wide application of internet technology have strongly promoted the development of commercial applications and scientific applications in the network environment, such as grid networks [1, 2, 9] and cellular networks [18, 19]. The cellular network is a mobile communication hardware architecture that divides the service of mobile phones into small regular hexagonal sub-areas, and each cell has a base station, forming a structure that resembles a "cellular" structure. Therefore, this mobile communication method is called cellular mobile communication method, and its structure can save equipment construction costs. The grid networks were developed to support large-scale scientific collaborative work.

Based on the importance of cellular networks and grid networks, it is natural to consider the disconnection colorings of them.

Consider a (planar, infinite) lattice of congruent regular hexagons (quadrangle) and a cycle C on it. Then the part of the hexagonal (quadrangle) lattice which lies in the interior of C and the cycle C itself, forms a *cellular networks* (*grid networks*) G [11]. We call the C the boundary of the network G. Denote by $E(G) - C$ the inner links of G. Obviously, the cellular networks and grid networks are 2-connected.

This paper is organized as follows. In Section 2, we obtain the (strong) rainbow disconnection numbers of cellular networks and grid networks. In Section 3, we give the rainbow node-disconnection numbers of cellular networks and grid networks. In Section 4, we present the proper and monochromatic disconnection numbers of cellular networks and grid networks.

2. Their rd-numbers and srd-numbers

For two distinct nodes u and v of G, let $\lambda_G(u, v)$ (or simply $\lambda(u, v)$ when the network G is clear from the context) denote the minimum number of links in a link-cut F such that u and v lie in different components of $G - F$. The minimum cardinality of a link-cut of G is the *link-connectivity* of G, denoted by $\lambda(G)$.

Lemma 2.1 [8] *If G is a nontrivial connected network, then*

$$\lambda(G) \leq \lambda^+(G) \leq rd(G) \leq \chi'(G) \leq \Delta(G) + 1,$$

where the upper link-connectivity $\lambda^+(G)$ is defined by $\lambda^+(G) = \max\{\lambda(u, v) : u, v \in V(G)\}$.

Lemma 2.2 [8] *Let G be a nontrivial connected network. Then $rd(G) = 2$ if and only if each block of G is either K_2 or a cycle and at least one block of G is a cycle.*

Theorem 2.3 *Let G be a cellular network with the number of hexagons h. Then*

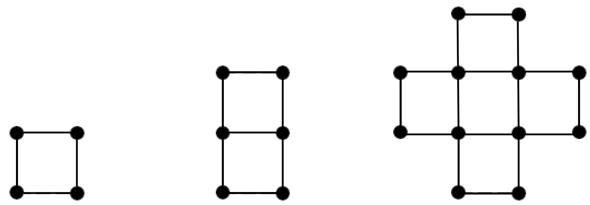
$$\mathrm{rd}(G) = \begin{cases} 2, & \text{if } h = 1, \\ 3, & \text{if } h \geq 2. \end{cases}$$

Proof. If $h = 1$, then $G = C_6$, so $\mathrm{rd}(G) = 2$ by Lemma 2.2. If $h \geq 2$, there exist two nodes u, v of G

satisfying $\lambda(u, v) = 3$. Moreover, we have $\chi'(G) = \Delta(G) = 3$ since G is a bipartite network. Hence, we get $\mathrm{rd}(G) = 3$ by Lemma 2.1.

Theorem 2.4 *Let G be a grid network. Then (see Figure 1)*

$$\mathrm{rd}(G) = \begin{cases} 2, & \text{if } G = G_1, \\ 3, & \text{if } H \subseteq G \text{ and } H \cong G_2, \text{ but no } H \subseteq G \text{ and } H \cong G_3, \\ 4, & \text{if } H \subseteq G \text{ and } H \cong G_3. \end{cases}$$



G_1 G_2 G_3

Figure 1: Grid networks in Theorem 2.4.

Proof. If $G = G_1$, then $\mathrm{rd}(G) = 2$. If $G_3 \subseteq G$, then $\lambda^+(G) = 4$. Moreover, since G is a bipartite network we have $\mathrm{rd}(G) \leq \chi'(G) = \Delta(G) = 4$. Hence, $\mathrm{rd}(G) = 4$.

Suppose that G has a subnetwork that is isomorphic to G_2, but no subnetwork that is isomorphic to G_3. Then we get $\mathrm{rd}(G) \geq 3$ by Lemma 2.1 since $\lambda^+(G) = 3$. It remains to prove that there exists an rd-coloring of G using 3 colors. First, we give two observations.

1. For any two nodes x and y of G with $d(x) = d(y) = 4$, if there has no a parallel 2 (3)-link-cut between x and y, then we can find a 3-link-cut $C(x, y)$ of x, y in G (see Figure 2).

2. For such two different 3-link-cuts in G, they have at most one common link in G, which ensures that there exists a coloring using colors [3] so that each 3-link-cut (like $C(x, y)$ in Figure 2) is rainbow.

We now divide these link-cuts into some families of link-cut: if two link-cuts belong to the same family, then one can find the other link-cut by link transitivity. Let G^* be the network obtained by deleting all such 3-link-cuts (like $C(x, y)$ in Figure 2) of G. Note that each nontrivial block of G^* is a subnetwork of $G_{3,i}$ ($i \geq 3$). We first assign a coloring c_0 for one component of G^*, say H_0, using colors [3] so that each set of links incident with a node of degree less than 4 and parallel 2 (3)-link-cuts in G are rainbow. Then, we color a family of link-cuts connected to the network H_0 so that each link-cut is a rainbow and each node is proper except for the nodes of degree 4 in G, and use H_1 to denote the new colored network. Furthermore, we colored other component of G^* connected with network H_1 and ensure that each node of H_1 is proper except the nodes of degree 4 in G and all parallel 2 (3)-link-cuts in G are rainbow. Repeatedly, we extent the coloring c_0 to a coloring c of G using colors [3] so that each parallel 2 (3)-link-cut and each set of links incident with a node of degree less than 4 in G is rainbow.

Now we can verify that the c is an rd-coloring of G. For any two nodes u, v of G, if there exists a node

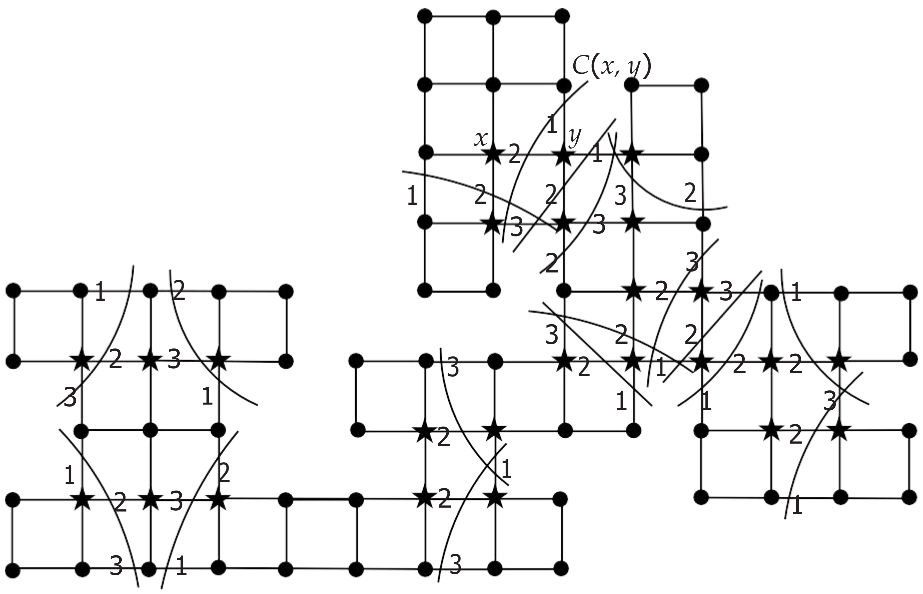


Figure 2: A network used in the proof of Theorem 2.4.

with degree less than 4, without loss of generality, say u, then the set E_u of links incident with node u is a u-v rainbow link-cut. If $d(u) = d(v) = 4$ and there has a parallel u-v 2 (3)-link-cut, then it is a u-v rainbow link-cut. If $d(u) = d(v) = 4$ and there has no parallel u-v 2 (3)-link-cut, then the $C(u, v)$ (like $C(x, y)$ in Figure 2) in network G is a u-v rainbow link-cut in G.

Furthermore, we study the strong rainbow disconnection numbers of cellular networks.

A *trivial link-cut S* of G is a link-cut incident with a node.

Lemma 2.5 [5] *If G is a connected network with link-connectivity* $\lambda(G)$*, upper link-connectivity* $\lambda^+(G)$ *and number* $e(G)$ *of links, then*

$$\lambda(G) \leq \lambda^+(G) \leq \mathrm{rd}(G) \leq \mathrm{srd}(G) \leq e(G). \qquad (1)$$

Lemma 2.6 [17] *A 3-connected cubic plane network G is* 4*-face-colorable if and only it is* 3*-link colorable, i.e.,* $\chi'(G) = 3$.

Lemma 2.7 [4] *A cube network G is* 3*-connected if and only if G is* 3*-link-connected.*

Lemma 2.8 [5] *Let G be a nontrivial connected network. Then srd*$(G) = 2$ *if and only if rd*$(G) = 2$.

Theorem 2.9 *Let G be a cellular network with the number of hexagons h. Then*

$$\mathrm{srd}(G) = \begin{cases} 2, & \text{if } h = 1, \\ 3, & \text{if } h \geq 2. \end{cases}$$

Proof. If $h = 1$, then $G = C_6$. By Lemmas 2.2 and 2.8, we have $\mathrm{srd}(G) = 2$. If $h \geq 2$, there exist two nodes u, v of G satisfying $\lambda(u, v) = 3$, so $\mathrm{srd}(G) \geq 3$ by Lemma 2.5. Now we define two operations o and O as follows.

$$o(\{G\}) = \begin{cases} \{G/V(C_1), G/V(C_2)\}, & \text{if } G \text{ has a nontrivial 2-link-cut } S \\ & \text{and } G \setminus S = C_1 \cup C_2, \\ \{G\}, & \text{otherwise.} \end{cases}$$

$$O(\{G_1, G_2, \cdots, G_p\}) = \cup_{i=1}^{p} o(\{G_i\}).$$

Since the network is split into two pieces when we do the operation, then the operation cannot last endlessly. Hence, there exists a integer r such that $O^r(\{G\}) = O^{r+1}(\{G\})$. Finally, we get a finite sequence of link-colored cubic networks H = $\{H_1, H_2, ,\cdots, H_q\}$, where q is a positive integer. Note that the operation does not appear multilinks, and each network of $\{H_1, H_2, \cdots ,H_q\}$ is planar. For each planar network $H \in$ H, we can construct a 3-link-connected 3-regular planar network H'. By the above operation, we know that each 2-link-cut in H is trivial and lies on the boundary of network H. Let h be the number of nodes with degree 2. Use h to denote the number of trivial 2-link-cuts in network H and give all nodes with degree 2 a clockwise label using $\{v_i : i \in [h]\}$. If $h \equiv 0$ (mod 3), then we add $h/3$ nodes, and make each node connect with 3 adjacent 2-degree nodes in H (starting from the node with degree 2 labeled 1, connect the links in turn clockwise, the same below); if $h \equiv 1$ (mod 3), we add $\lfloor h/3 \rfloor - 1$ nodes, and make each node connect to the 3 adjacent nodes with degree 2 in H. For the remaining 4 nodes with degree 2, we add two links $v_{h-3}v_{h-2}$ and $v_{h-1}v_h$; if $h \equiv 2$ (mod 3), we add $\lfloor h/3 \rfloor$ nodes, and make each node connect to the 3 adjacent nodes with degree 2 in H, and then add a link between the remaining two nodes with degree 2. It is easy to verify that the network H' is a 3-link-connected 3-regular plane network. By Lemmas 2.6 and 2.7, it implies that H' is 3-link-colorable. Then each network H is 3-link-colorable, and we use color set [3] to assign a proper link-coloring to each network in H. Then we perform the inverse operation of the shrinking operation. Assume that F_1 and F_2 are two proper link-colored networks obtained by shrinking the non-trivial 2-link-cut $\{e_1, e_2\}$ of network F, and let c_1 and c_2 be colorings of networks F_1 and F_2 using colors [3], respectively. Obviously, $c_1(e_1) \neq c_1(e_2)$ and $c_2(e_1) \neq c_2(e_2)$. Now we exchange the colors $c_1(e_1)$ and $c_2(e_1)$, and colors $c_1(e_2)$ and $c_2(e_2)$ in F_1 such that the new coloring c_1' of F_1 satisfies $c_1'(e_1) = c_2(e_1)$ and $c_1'(e_2) = c_2(e_2)$. Obviously, c_1' is still a proper link-coloring of the network F_1 using the color set [3]. Then we can get a link-coloring c_0 of network F: let $c_0(e) = c_1'(e)$, if e $\in F_1$; let $c_0(e) = c_2(e)$, if $e \in F_2$. Obviously, the c_0 is a proper link-coloring of network F. Continue to do this, and finally we get a proper link-coloring c of the network G using the color set [3].

Now we verify that the link-coloring c of G is a strong rainbow disconnection coloring of the network G. Let u and v be two nodes of G, and assume that $d(u) \leq d(v)$. If $d(u) = 2$, then the link set E_u is a minimum u-v link-cut of G and rainbow, so the link set E_u is a rainbow minimum u-v link-cut of G; if $d(u) = d(v) = 3$ and $\lambda(u, v) = 3$, then the link set E_u is a minimum u-v link-cut of G and rainbow, so the link set E_u is a rainbow minimum u-v link-cut of G; if $d(u) = d(v) = 3$ and $\lambda(u, v) = 2$. By the contraction operation, we get that u and v belong to different connected components in H (otherwise, suppose that both u and v belong to a connected component H of H. Since $\lambda(u, v) = 2$, and the shrinking operation does not change the link connectivity of u, v, there is still a nontrivial 2-link-cut between u and v. This is a contradiction with our operation). Therefore, there exists a rainbow 2-link-cut $C(u, v)$ between u and v by the process of operation and coloring, and the $C(u, v)$ is a rainbow minimum u-v link-cut of G. Hence, srd(G) ≤ 3.

Moreover, we conjecture that the strong rainbow disconnection numbers of grid networks are equal to the rainbow disconnection numbers of grid networks.

Conjecture 2.10 *Let G be a grid network (see Figure 1). Then*

$$\mathrm{srd}(G)=\begin{cases}2, & \textit{if } G=G_1,\\ 3, & \textit{if } H\subseteq G \textit{ and } H\cong G_2, \textit{ but no } H\subseteq G \textit{ and } H\cong G_3,\\ 4, & \textit{if } H\subseteq G \textit{ and } H\cong G_3.\end{cases}$$

3. THEIR RND-NUMBERS

Next, we study the node-version of rainbow disconnection coloring.

Lemma 3.1 [3] *If C_n is a cycle of order $n \geq 3$, then $rnd(C_n) = 2$.*

Lemma 3.2 [3] *If G is a nontrivial connected network and H is a connected subnetwork of G, then $rnd(H) \leq rnd(G)$.*

Lemma 3.3 [3] *Let G be a nontrivial connected network of order n. Then $\kappa(G) \leq \kappa^+(G) \leq rnd(G) \leq n$.*

Theorem 3.4 *Let G be a cellular network with the number of hexagons h. Then*

$$\mathrm{rnd}(G)=\begin{cases}2, & \textit{if } h=1,\\ 3, & \textit{if } h\geq 2.\end{cases}$$

Proof. If $h = 1$, then we have $\mathrm{rnd}(G) = 2$ by Lemma 3.1. If $h \geq 2$, we select the common link of some two hexagons, say v_1v_2. We have $\mathrm{rnd}(G) \geq \kappa_G(v_1, v_2) \geq 3$. For the nodes of G, assign column numbers according to the order in which they appear from left to right in the lattice shown in the figure 3. For example, the nodes in the same column which appear first are labeled column 1. Now we give a node-coloring c of G using three colors. For the nodes in the column j of network G, if $j \equiv 1 \pmod 3$, then color them by 1; if $j \equiv 2 \pmod 3$, then color them by 2; if $j \equiv 0 \pmod 3$, then color them by 3. Let v be any node of network G. Assume that v is in the column i of G. If $d_G(v) = 2$, then the neighbors of v are in columns $i-1$, $i+1$ or i, $i+1$ or $i-1$, i. Since the column labels of the neighbors are different modulo 3, we have $N_G(v)$ is rainbow. If $d_G(v) = 3$, then the neighbors of v are in columns $i-1$, i, $i+1$, respectively. Since $i-1$, i, $i+1$ are pairwise different modulo 3, we have that $N_G(v)$ is rainbow.

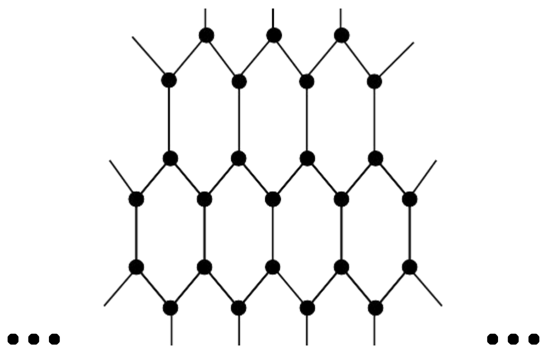


Figure 3: A (planar, infinite) lattice of congruent regular hexagons.

Let x and y be two nodes of network G. If x, y are adjacent, then $N_G(x) \setminus \{y\}$ is an x-y rainbow node-cut. If x, y are nonadjacent, then $N_G(x)$ is an x-y rainbow node-cut. So c is a rainbow node-disconnection coloring of network G. We obtain $\mathrm{rnd}(G) \leq 3$.

The Cartesian product $G \square H$ of two internal disjoint networks G and H is the network with node set $V(G) \times V(H)$, where (u, v) is adjacent to (w, x) in $G \square H$ if and only if either $u = w$ and $vx \in E(H)$ or $uw \in E(G)$ and $v = x$. The $m \times n$ grid network $G_{m,n} = P_m \square P_n$ consists of m horizontal paths P_n and n vertical paths P_m.

Lemma 3.5 *For $n \geq 3$, $rnd(G_{3,n}) = 3$.*

Proof. Define a node-coloring c: $V(G_{3,n}) \to [3]$ of $G_{3,n}$. Let $c(x_{1,j}) = 1$ for $j \equiv 1, 2 \pmod 4$ and $c(x_{1,j}) = 2$ for $j \equiv 0, 3 \pmod 4$. We color the second row using color 3. Let $c(x_{3,j}) = 2$ for $j \equiv 1, 2 \pmod 4$ and $c(x_{3,j}) = 1$ for $j \equiv 0, 3 \pmod 4$. We show that c is a rainbow node-disconnection coloring of $G_{3,n}$. Let $x_{p,q}$ and $x_{s,\ell}$ be two nodes of network $G_{3,n}$, where $p \leq s$.

If $p = 1$, then $N_{G_{3,n}}(x_{p,q})$ is rainbow. So when $x_{p,q}$ and $x_{s,l}$ are nonadjacent, $N_{G_{3,n}}(x_{p,q})$ is an $x_{p,q}$-$x_{s,\ell}$ rainbow node-cut; when $x_{p,q}$ and $x_{s,\ell}$ are adjacent, $N_{G_{3,n}}(x_{p,q})\backslash\{x_{s,\ell}\}$ is an $x_{p,q}$-$x_{s,\ell}$ rainbow node-cut. If $s = 3$, then $N_{G_{3,n}}(x_{s,\ell})$ is rainbow. Similarly, there is a rainbow node-cut between $x_{p,q}$ and $x_{s,\ell}$.

Now consider $p = s = 2$. Suppose that $q < \ell$. If $x_{p,q}$ and $x_{s,\ell}$ are nonadjacent, $\{x_{p-1,q}, x_{p,q+1}, x_{p+1,q}\}$ is an $x_{p,q}$-$x_{s,\ell}$ rainbow node-cut. If $x_{p,q}$ and $x_{s,\ell}$ are adjacent, $\{x_{p-1,q}, x_{p+1,q}\}$ is an $x_{p,q}$-$x_{s,\ell}$ rainbow node-cut.

So we have $rnd(G_{3,n}) \leq 3$. Since $\kappa(x_{1,2}, x_{2,2}) = 3$, we have $rnd(G) \geq \kappa(x_{1,2}, x_{2,2}) = 3$ by Lemma 3.3. □

Lemma 3.6 *For $4 \leq m \leq n$, $rnd(G_{m,n}) = 4$.*

Proof. Define a node-coloring c of $G_{m,n}$: $V(G_{m,n}) \to Z_4$. Let $c(x_{i,1}) = i \pmod 4$, $c(x_{i,2}) = c(x_{i,3}) = i + 2 \pmod 4$ and $c(x_{i,4}) = i \pmod 4$. Other remaining columns repeat the coloring of first four columns.

Let u be a node of $G_{m,n}$ and $N_r(u)$ ($N_c(u)$) denote the neighbors of u in the same row (column). Assume that $c(u) = a$. If $|N_r(u)| = 2$, then two nodes of $N_r(u)$ are assigned a and $a + 2$ respectively; if $|N_r(u)| = 1$, then it is assigned a or $a + 2$. If $|N_c(u)| = 2$, then two nodes of $N_c(u)$ are assigned $a - 1$ and $a + 1$ respectively; if $|N_r(u)| = 1$, then it is assigned $a - 1$ or $a + 1$. Thus, $N_{G_{m,n}}(u)$ is rainbow.

For any two nonadjacent nodes x and y of $G_{m,n}$, $N_{G_{m,n}}(x)$ is an x-y rainbow node-cut. For any two adjacent nodes x and y of $G_{m,n}$, $N_{G_{m,n}}(x) \setminus \{y\}$ is an x-y rainbow node-cut. The coloring c is a rainbow node-disconnection coloring of $G_{m,n}$,. Hence, rnd(G) ≤ 4. On the other hand, $\kappa(x_{2,2}, x_{3,3}) = 4$. It follows by Lemma 3.3 that rnd($G_{m,n}$,) ≥ $\kappa(x_{2,2}, x_{3,3}) = 4$.

For a node-cut S of G, we denote the connected components of $G - S$ by $G_1, G_2, \cdots, G_s$. Then we add S to these components and get networks $G[V(G_1) + S], G[V(G_2) + S], \cdots, G[V(G_s) + S]$. This operation is called that we *split* the node-cut S.

If the nodes of a 2-node-cut of G are adjacent, then we say the 2-node-cut is an *adjacent 2-node-cut.*

Theorem 3.7 *Let G be a grid network. Then (as shown in Figure 4)*

$$\mathrm{rnd}(G) = \begin{cases} 2, & if\ G = G_1, \\ 3, & if\ G_2 \subseteq G\ and\ G_3, G_4 \nsubseteq G, \\ 4, & if\ G_3 \subseteq G\ or\ G_4 \subseteq G. \end{cases}$$

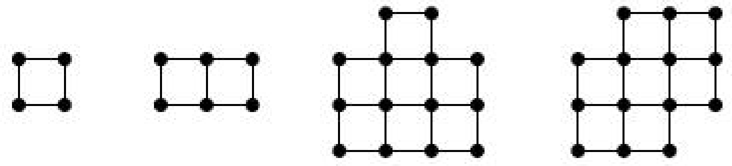


Figure 4: Grid networks in Theorem 3.7.

Proof. If $G = G_1$, then we have rnd(G) = 2 by Lemma 3.1. If $G_3 \subseteq G$ or $G_4 \subseteq G$, then rnd(G) $\geq \kappa^+(G) \geq$ 4 by Lemma 3.3. Since G is the subnetwork of some grid network $G_{m,n}$, we have rnd(G) $\leq$ rnd($G_{m,n}$) = 4 by Lemmas 3.2 and 3.6.

Now consider $G_2 \subseteq G$ and $G_3, G_4 \not\subseteq G$.

We have rnd(G) $\geq \kappa^+(G) \geq \kappa^+(G_2) \geq 3$. If $G = G_{3,n}$, then rnd(G) = 3 by Lemma 3.5.

If $G \neq G_{3,n}$, then there exists an adjacent 2-node-cut. We split all adjacent 2-node-cuts. Then we can get networks $H_1, H_2, \cdots, H_\ell$. Obviously, each H_i is a 4-cycle or $G_{3,n}$. Then we do the following operations.

1. Select the network H_1 and color H_1 using rnd-coloring c_1. Let $H = H_1$ and $c_H = c_1$.

2. Select the network H_i which has a common adjacent 2-node-cut S with network H and color H_i using rnd-coloring c_i.

3. Let $H = H \cup H_i$ and $c_H = c_H + c_i$. If H and G are not isomorphic, then return to step 2.

The rnd-colorings c_i ($i \in [\ell]$) are as follows.

c_1: If H_1 is a 4-cycle, then we assign color 1 to two adjacent nodes and assign 2,3 to the remaining two nodes. If H_1 is $G_{3,n}$, then we color it using the same coloring as Lemma 3.5.

c_i ($i \in \{2, 3, \cdots, \ell\}$): Assume that $S = \{u, v\}$. Let $c_i(u) = c_H(u)$ and $c_i(v) = c_H(v)$.

If H_i is a 4-cycle, we denote the 4-cycle containing link uv in H by C_i. We color the neighbors of u and v in H_i using the colors different from $N_{C_i}(u)$ and $N_{C_i}(v)$ respectively. Obviously, we finish the color of H_i.

Next, consider $H_i = G_{3,n}$. Obviously, u, v have at least one node with degree four in G and degree two in H. Without loss of generality, assume that $d_G(v) = 4$ and $d_H(v) = 2$. Let $N_H(v) = \{u, v_1\}$. We use two stages to color H_i.

- If $d_G(u) = 3$, then color the neighbor of u in H_i such that $N_G(u)$ is rainbow.

  If $d_G(u) = 4$, then $d_{H_i}(u) = 2$. Let $N_{H_i}(u) = \{v, u_1\}$. When $\{u, v, v_1\}$ is rainbow, let $c_i(u_1) = c_i(u)$; otherwise, color u_1 such that $\{u_1, v_1, u\}$ is rainbow.

- Color the remaining nodes of H_i according to Figure 5.

In first stage, we color three nodes of H_i. No matter how we color it, the colors of three nodes have three cases as shown in Figure 5, where the three nodes are marked by stars and $\{a, b, c\} = \{1, 2, 3\}$ are three different colors.

In second stage, for the networks in Figure 5, other columns of H', H'' and H''' repeat the colors of columns 1-4.

Similar to the proof of Lemma 3.5, we can get that c_i is an rnd-coloring of H_i for $i \in [\ell]$.

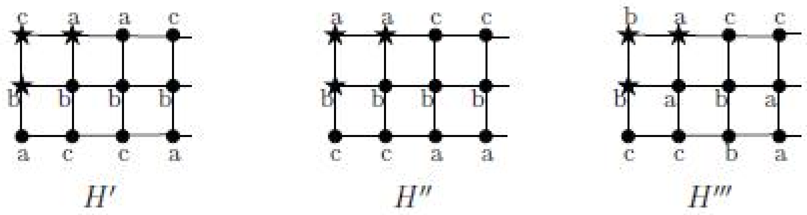


Figure 5: Three node-colorings of $G_{3,n}$.

Now we claim that the node-coloring of $H \cup H_i$ is an rnd-coloring. Based on the process of coloring, the neighborhoods of nodes with degree less than four are rainbow. So we only need to consider two nodes with degree four.

Let x, y be two nodes of $H \cup H_i$ with degree four. Assume that R_H is an x-y rainbow node-cut of H under c_H . Let R_i *be* an x-y rainbow node-cut of H_i under c_i. Consider $\{x, y\} = \{u, v\}$. Then $\{u_1, v_1, v\}$ or $\{u_1, v_1, u\}$ is an x-y rainbow node-cut of $H \cup H_i$.

Consider $\{x, y\} \neq \{u, v\}$. If $x, y \in V(H)$, then R_H is an x-y rainbow node-cut of $H \cup H_i$. If $x, y \in V(H_i)$, then R_i is an x-y rainbow node-cut of $H \cup H_i$. If $x \in V(H) \setminus \{u, v\}$, $y \in V(H_i) \setminus \{u, v\}$ or $x \in V(H_i) \setminus \{u, v\}$, $y \in V(H) \setminus \{u, v\}$, then $\{v, u_1\}$ is an x-y rainbow node-cut of $H \cup H_i$.

So the above operations keep new network $H = H \cup H_i$ rainbow node-disconnected. Therefore, rnd(G) = 3.

4. THEIR PD-NUMBERS AND MD-NUMBERS

Furthermore, we obtain the proper and monochromatic disconnection numbers of cellular networks and grid networks.

Observation 4.1 *Let G be a cellular network. Then pd(G) = 1.*

Observation 4.2 *Let G be a grid network. Then* $pd(G) = 1$.

Theorem 4.3 *Let G be a cellular network with the number of hexagons h, the number of inner links m and the boundary C. Then* $md(G) = 3h - m = |C|/2$.

Proof. Observe that each color appears at least 2 times in an md-coloring, so one hexagon has at most 3 colors. If two hexagons have a common link, then the two hexagon use at most 5 colors under an md-coloring in G. Then an md-coloring of G has at most $3h - m$ colors since G has m pairs of hexagons with a common link. Namely, md(G) $\leq 3h - m$.

Now we give a coloring f of G. First, we give a link partition for G. For two adjacent hexagons H_1, H_2, let e be the common link of H_1, H_2. Then there are opposite links e_1 and e_2 of e in H_1 and H_2, respectively. If e_1 or e_2 is not a bounded link, then we continue to find the opposite link of e_1 or e_2 in other hexagon, and call all these opposite links a relative link set, denoted by M_i, ($i \in [t]$). Observe that $\mathrm{E(G)} = \bigcup_{i=1}^{t} M_i$ and $t = |C|/2$. Next, for each $i \in [t]$, we assign color i to all links of M_i, therefore $|f| = t$. Moreover, we get $|f| = 3h - m$ since $|C| + m = 6h - m$. It is easy to verify that the coloring f is an md-coloring of G. Hence, $\mathrm{md}(G) = 3h - m$.

Theorem 4.4 *Let G be a grid network with the number of quadrangle h, the number of inner links m and the boundary C. Then* $md(G) = 2h - m = |C|/2$.

The proof of Theorem 4.4 is similar to the argument of Theorem 4.3.

5. Conclusions

In this paper, we get the exact values of the rainbow (node-)disconnection numbers, proper and monochromatic disconnection numbers of cellular networks and grid networks, respectively, and we conjecture that the strong rainbow disconnection numbers of grid networks are equal to the rainbow disconnection numbers of grid networks.

Acknowllinkments

The authors are very grateful to the reviewers and editor for their useful suggestions and comments, which helped to improve the presentation of the paper. This work was supported by NSFC No.11871034. X. Li is the corresponding author.

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