ON ANNIHILATOR OF INTUITIONISTIC FUZZY SUBSETS OF MODULES

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ABSTRACT

In the theory of rings and modules there is a correspondence between certain ideals of a ring R and submodules of an R-module that arise from annihilation. The submodules obtained using annihilation, which correspond to prime ideals play an important role in decomposition theory. In this paper, we attempt to intuitionistic fuzzify the concept of annihilators of subsets of modules. We investigate certain characterization of intuitionistic fuzzy annihilators of subsets of modules. Using the concept of intuitionistic fuzzy annihilators, intuitionistic fuzzy prime submodules and intuitionistic fuzzy annihilator ideals are defined and various related properties are established.

KEYWORDS

Intuitionistic fuzzy submodule, intuitionistic fuzzy prime submodule, intuitionistic fuzzy ideal, intuitionistic fuzzy annihilator, semiprime ring.

1. INTRODUCTION

The concept of intuitionistic fuzzy sets was introduced by Atanassov [1], [2] as a generalization to the notion of fuzzy sets given by Zedah [16]. Biswas was the first to introduce the intuitionistic fuzzification of the algebraic structure and developed the concept of intuitionistic fuzzy subgroup of a group in [5]. Hur and others in [8] defined and studied intuitionistic fuzzy subrings and ideals of a ring. In [7] Davvaz et al. introduced the notion of intuitionistic fuzzy submodules which was further studied by many mathematicians (see [4], [9], [12], [13], [14]).

The correspondence between certain ideals and submodules arising from annihilation plays a vital role in the decomposition theory and Goldie like structures (see [6]). A detailed study of the fuzzification of this and related concepts can be found in [10], [11] and [15]. Intuitionistic fuzzification of such crisp sets leads us to structures that can be termed as intuitionistic fuzzy prime submodules. In this paper, we attempt to define annihilator of an intuitionistic fuzzy subset of a module using the concept of residual quotients and investigate various characteristic of it. This concept will help us to explore and investigate various facts about the intuitionistic fuzzy aspects of associated primes, Godlie like structures and singular ideals.

2. PRELIMINARIES

Throughout this section, R is a commutative ring with unity 1, $1 \neq 0$, M is a unitary R-module and θ is the zero element of M. The class of intuitionistic fuzzy subsets of X is denoted by IFS(X).

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Definition (2.1)[4] Let R be a ring. Then A ∈ IFS(R) is called an intuitionistic fuzzy ideal of R if for all x , $y \in$ R it satisfies

$$(i)\ \mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)\ ,\ \ \nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y);$$
$$(ii)\ \mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)\ \ ,\ \ \nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y).$$

The class of intuitionistic fuzzy ideals of R is denoted by IFI(R).

Definition (2.2)[4] An intuitionistic fuzzy set A = (μ_A , ν_A) of an R-module M is called an intuitionistic fuzzy submodule (IFSM) if for all x , $y \in$ M and $r \in$ R , we have

$$(i)\ \mu_A(\theta) = 1\ ,\ \ \nu_A(\theta) = 0;$$
$$(ii)\ \mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)\ ,\ \ \nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y);$$
$$(iii)\ \mu_A(rx) \geq \mu_A(x)\ ,\ \ \nu_A(rx) \leq \nu_A(x).$$

The class of intuitionistic fuzzy submodules of M is denoted by IFM(M).

Definition (2.3)[2, 12] Let α, β ∈ [0, 1] with α + β ≤ 1. An intuitionistic fuzzy point, written as $x_{(\alpha,\beta)}$, is defined to be an intuitionistic fuzzy subset of X, given by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha,\beta) & ;\ if\ \ y = x \\ (0,1) & ;\ if\ \ y \neq x \end{cases}.$$ We write $x_{(\alpha,\beta)} \in A$ if and only if $x \in C_{(\alpha,\beta)}(A)$,

where $C_{(\alpha,\beta)}(A) = \{x \in X :\ \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta\}$ is the $(\alpha,\beta)-cut$ set (crisp set) of the intuitionistic fuzzy set A in X.

Definition (2.4)[9] Let M be an R-module and let A, B ∈ IFM(M). Then the sum A + B of A and B is defined as

$$\mu_{A+B}(x) = \begin{cases} \mu_A(\mathrm{y}) \wedge \mu_B(\mathrm{z}) & ;\ if\ \ x = y+z \\ 0 & ;\ \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{A+B}(x) = \begin{cases} \nu_A(\mathrm{y}) \vee \nu_B(\mathrm{z}) & ;\ if\ \ x = y+z \\ 1 & ;\ \text{otherwise} \end{cases}$$

Then, A + B∈ IFM(M).

Definition (2.5) Let M be an R-module and let A∈ IFS(R) and B∈ IFM(M). Then the product AB of A and B is defined as

$$\mu_{AB}(x) = \begin{cases} \mu_A(r) \wedge \mu_B(m) & ;\ if\ \ x = rm \\ 0 & ;\ \text{otherwise} \end{cases} \text{ and } \nu_{AB}(x) = \begin{cases} \nu_A(r) \vee \nu_B(m) & ;\ if\ \ x = rm \\ 1 & ;\ \text{otherwise} \end{cases} ,\ r \in R,\ m \in M.$$

Clearly, AB∈ IFM(M).

Definition (2.6) [9, 12] Let M be an R-module and let A, B ∈ IFM(M). Then the product AB of A and B is defined as

$$\mu_{AB}(x) = \begin{cases} \mu_A(y) \wedge \mu_B(z) & ;\ if\ \ x = yz \\ 0 & ;\ \text{otherwise} \end{cases} \text{ and } \nu_{AB}(x) = \begin{cases} \nu_A(y) \vee \nu_B(z) & ;\ if\ \ x = yz \\ 1 & ;\ \text{otherwise} \end{cases} ,\text{where } y, z \in M.$$

Definition (2.7) [3] An P∈ IFI(R) is called an intuitionistic fuzzy prime ideal of R if for any A, B ∈ IFI(R) the condition AB ⊆ P implies that either A ⊆ P or B ⊆ P.

Definition (2.8) Let X be a non empty set and A ⊂ X. Then an intuitionistic fuzzy set $\chi_A = \left(\mu_{\chi_A}, \nu_{\chi_A}\right)$ is called an intuitionistic fuzzy characteristic function and is defined as

$$\mu_{\chi_A}(x) = \begin{cases} 1 & ; \text{ if } x \in A \\ 0 & ; \text{ if } x \notin A \end{cases} \quad \text{and} \quad \nu_{\chi_A}(x) = \begin{cases} 0 & ; \text{ if } x \in A \\ 1 & ; \text{ if } x \notin A \end{cases}.$$

Definition (2.9) Let M be R-module and χ_θ is an IFS on M defined as $\chi_\theta(x) = \left(\mu_{\chi_\theta}(x), \nu_{\chi_\theta}(x)\right)$,

$$\text{where } \mu_{\chi_\theta}(x) = \begin{cases} 1 & ; \text{ if } x = \theta \\ 0 & ; \text{ if } x \neq \theta \end{cases} \quad \text{and} \quad \nu_{\chi_\theta}(x) = \begin{cases} 0 & ; \text{ if } x = \theta \\ 1 & ; \text{ if } x \neq \theta \end{cases}$$

and χ_0 and χ_R are IFSs on R defined by

$\chi_0(r) = \left(\mu_{\chi_0}(r), \nu_{\chi_0}(r)\right)$ and $\chi_R(r) = \left(\mu_{\chi_R}(r), \nu_{\chi_R}(r)\right)$, where

$$\mu_{\chi_0}(r) = \begin{cases} 1 & ; \text{ if } r = 0 \\ 0 & ; \text{ if } r \neq 0 \end{cases}; \ \nu_{\chi_0}(r) = \begin{cases} 0 & ; \text{ if } r = 0 \\ 1 & ; \text{ if } r \neq 0 \end{cases} \text{ and } \mu_{\chi_R}(r) = 1 \ ; \ \nu_{\chi_R}(r) = 0, \ \forall \ r \in R.$$

Theorem (2.10) Let $x \in$ R and α , β ∈ (0,1] with α + β ≤ 1. Then $< x_{(\alpha,\beta)} > = (\alpha,\beta)_{<x>}$,

where $(\alpha,\beta)_{<x>}(y) = \begin{cases} (\alpha,\beta) & ; \text{ if } y \in < x > \\ (0,1) & ; \text{ if } y \notin < x > \end{cases}$ is called the $(\alpha,\beta)-level(or\ cut\ set)$ intuitionistic fuzzy ideal corresponding to $< x >$.

Proof. Case(i) When $y \in < x >$ and let $y = x^n$, for some positive interget n, then

$\mu_{(\alpha,\beta)_{<x>}}(y) = \alpha = \mu_{x_{(\alpha,\beta)}}(x) \leq \mu_{x_{(\alpha,\beta)}}(x^n) = \mu_{x_{(\alpha,\beta)}}(y)$ and

$\nu_{(\alpha,\beta)_{<x>}}(y) = \beta = \nu_{x_{(\alpha,\beta)}}(x) \geq \nu_{x_{(\alpha,\beta)}}(x^n) = \nu_{x_{(\alpha,\beta)}}(y)$

Case(ii) When $y \notin < x >$, then

$\mu_{(\alpha,\beta)_{<x>}}(y) = 0 = \mu_{x_{(\alpha,\beta)}}(y)$ and $\nu_{(\alpha,\beta)_{<x>}}(y) = 1 = \nu_{x_{(\alpha,\beta)}}(y)$.

Thus in both the cases we find that $(\alpha,\beta)_{<x>} \subseteq x_{(\alpha,\beta)}$.

Now $< x_{(\alpha,\beta)} > = \cap\left\{A : A \in IFI(\mathrm{R}) \text{ such that } x_{(\alpha,\beta)} \subseteq A\right\}$ implies that $(\alpha,\beta)_{<x>} = < x_{(\alpha,\beta)} >$.

3. ANNIHILATOR OF INTUITIONISTIC FUZZY SUBSET OF R-MODULE

Throughout this section, R is a commutative ring with unity 1, 1 ≠ 0, M is a unitary R-module and θ is the zero element of M.

Definition (3.1) Let M be a R-module and A∈ IFS(M), then the annihilator of A is denoted by ann(A) and is defined as: $ann(A) = \bigcup \{\mathrm{B} : \mathrm{B} \in IFS(R) \text{ such that } BA \subseteq \chi_\theta\}$.

Lemma (3.2) Let M be a R-module, then $ann(\chi_\theta) = \chi_R$.

Proof. Since $\chi_\theta \in IFS(\mathrm{M})$ and $\chi_R \in IFS(\mathrm{R})$, therefore, $\chi_R\chi_\theta \in IFS(\mathrm{M})$.
Also, $\chi_R\chi_\theta(x) = \left(\mu_{\chi_R\chi_\theta}(x),\ \nu_{\chi_R\chi_\theta}(x)\right)$, where
$\mu_{\chi_R\chi_\theta}(x) = \vee\{\chi_R(r) \wedge \chi_\theta(m):\ r \in R,\ m \in M,\ rm = x\}$ and
$\nu_{\chi_R\chi_\theta}(x) = \wedge\{\chi_R(r) \vee \chi_\theta(m):\ r \in R,\ m \in M,\ rm = x\}$.

$$Now,\ \mu_{\chi_R\chi_\theta}(x) = \vee\{\chi_R(r) \wedge \chi_\theta(m):\ r \in R,\ m \in M,\ rm = x\}$$
$$= \vee\{\chi_\theta(m):\ r \in R,\ m \in M,\ rm = x\}$$
$$= \begin{cases} 1 & ;\ if\ \ x = \theta \qquad [\because\ if\ \ x = \theta \Rightarrow one\ m = \theta] \\ 0 & ;\ if\ \ x \neq \theta \qquad [\because\ if\ \ x \neq \theta \Rightarrow m \neq \theta] \end{cases}$$

$$Also,\ \nu_{\chi_R\chi_\theta}(x) = \wedge\{\chi_R(r) \vee \chi_\theta(m):\ r \in R,\ m \in M,\ rm = x\}$$
$$= \wedge\{\chi_\theta(m):\ r \in R,\ m \in M,\ rm = x\}$$
$$= \begin{cases} 0 & ;\ if\ \ x = \theta \\ 1 & ;\ if\ \ x \neq \theta \end{cases}.$$

$Thus,\ \chi_R\chi_\theta(x) = \chi_\theta(x)$.
$So,\ \chi_R \subseteq \bigcup\{B:\ B \in IFS(R),\ \ B\chi_\theta \subseteq \chi_\theta\} = ann(\chi_\theta) \subseteq \chi_R$.
$Hence\ \ ann(\chi_\theta) = \chi_R$.

Lemma (3.3) Let M be a R-module and A∈IFS(M), then $\chi_0 \subseteq ann(A)$.

Proof. Now, $\mu_{\chi_0 A}(x) = \vee\{\mu_{\chi_0}(r) \wedge \mu_A(m):\ r \in R,\ m \in M,\ rm = x\}$
When $x \neq \theta \Rightarrow r \neq 0;\ \forall\ r \in R$, such that $rm = x$
$\Rightarrow\ \mu_{\chi_0}(r) = 0\ \forall\ r \in R$, such that $rm = x$. So, $\mu_{\chi_0 A}(x) = 0 = \mu_{\chi_\theta}(x)$.
When $x = \theta \Rightarrow \mu_{\chi_0 A}(\theta) \leq 1 = \mu_{\chi_\theta}(\theta)$. Thus, $\mu_{\chi_0 A}(x) \leq \mu_{\chi_\theta}(x)$.
Similarly, we can show that $\nu_{\chi_0 A}(x) \geq \nu_{\chi_\theta}(x)$. Therefore, $\chi_0 A \subseteq \chi_\theta$.
Hence $\chi_0 \subseteq \bigcup\{\mathrm{B}:\ \mathrm{B} \in IFS(R)$ such that $\mathrm{BA} \subseteq \chi_\theta\} = ann(A)$.

Lemma (3.4) Let M be a R-module and A, B∈IFS(M). If A ⊆ B, then
$ann(B) \subseteq ann(A)$.

Proof. Let A, B $\in IFS(\mathrm{M})$, $C \in IFS(\mathrm{R})$. Then $CA(x) = \left(\mu_C(x),\ \nu_{CA}(x)\right)$, where
$\mu_{CA}(x) = \vee\{\mu_C(r) \wedge \mu_A(m):\ r \in R,\ m \in M,\ rm = x\}$ and
$\nu_{CA}(x) = \wedge\{\nu_C(r) \vee \nu_A(m):\ r \in R,\ m \in M,\ rm = x\}$.
Now, $\mu_C(r) \wedge \mu_A(m) \leq \mu_C(r) \wedge \mu_B(m)$

$$\text{Therefore, } \mu_{CA}(x) = \vee\{\mu_C(r) \wedge \mu_A(m):\ r \in R,\ m \in M,\ rm = x\}$$
$$\leq \vee\{\mu_C(r) \wedge \mu_B(m):\ r \in R,\ m \in M,\ rm = x\}$$
$$= \mu_{CB}(x).$$

Similarly, we can show that $\nu_{CA}(x) \geq \nu_{CB}(x)$. Thus $CA \subseteq CB$.

So, $CB \subseteq \chi_\theta \Rightarrow CA \subseteq \chi_\theta$.

$\therefore \bigcup\{C : C \in IFS(R) \text{ such that } CB \subseteq \chi_\theta\} \subseteq \bigcup\{C : C \in IFS(R) \text{ such that } CA \subseteq \chi_\theta\}$

$\Rightarrow ann(\mathrm{B}) \subseteq ann(\mathrm{A})$.

Theorem (3.5) Let M be a R-module and A∈IFS(M). Then

$$ann(A) = \bigcup\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in [0,1] \text{ with } \alpha+\beta \leq 1 \text{ such that } r_{(\alpha,\beta)}A \subseteq \chi_\theta\}$$

Proof. We know that

$\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in [0,1] \text{ with } \alpha+\beta \leq 1 \text{ such that } r_{(\alpha,\beta)}A \subseteq \chi_\theta\} \in IFS(\mathrm{R})$

$\therefore \{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in [0,1] \text{ with } \alpha+\beta \leq 1 \text{ such that } r_{(\alpha,\beta)}A \subseteq \chi_\theta\}$

$\subseteq \{\mathrm{B} : \mathrm{B} \in IFS(R) \text{ such that } BA \subseteq \chi_\theta\}$

$\Rightarrow \bigcup\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in [0,1] \text{ with } \alpha+\beta \leq 1 \text{ such that } r_{(\alpha,\beta)}A \subseteq \chi_\theta\}$

$\subseteq \bigcup\{\mathrm{B} : \mathrm{B} \in IFS(R) \text{ such that } BA \subseteq \chi_\theta\} = ann(A)$.

Let $\mathrm{B} \in IFS(\mathrm{R})$ such that $\mathrm{BA} \subseteq \chi_\theta$.

Let $r \in \mathrm{R}$ and $\mathrm{B}(r) = (\alpha, \beta)$, i.e., $\mu_B(r) = \alpha$ and $\nu_B(r) = \beta$.

Now, $(r_{(\alpha,\beta)}A)(x) = (\mu_{r_{(\alpha,\beta)}A}(x), \ \nu_{r_{(\alpha,\beta)}A}(x))$, where

$$\begin{aligned}\mu_{r_{(\alpha,\beta)}A}(x) &= \vee\{\mu_{r_{(\alpha,\beta)}}(s) \wedge \mu_A(y) : s \in R, \ y \in M, \ sy = x\}\\ &\leq \vee\{\mu_B(r) \wedge \mu_A(y) : \ y \in M, \ ry = x\} \ [\because \mu_{r_{(\alpha,\beta)}}(s) \leq \mu_B(r) = \alpha]\\ &= \vee\{\mu_B(s) \wedge \mu_A(y) : s \in R, \ y \in M, \ sy = x\}\\ &= \mu_{BA}(x)\\ &\leq \mu_{\chi_\theta}(x)\end{aligned}$$

i.e., $\mu_{r_{(\alpha,\beta)}A}(x) \leq \mu_{\chi_\theta}(x), \ \forall \ x \in M$.

Similarly, we can show that $\nu_{r_{(\alpha,\beta)}A}(x) \geq \nu_{\chi_\theta}(x), \ \forall \ x \in M$. Thus, $r_{(\alpha,\beta)}A \subseteq \chi_\theta$.

So, $\mathrm{ann(A)} \subseteq \bigcup\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in [0,1] \text{ with } \alpha+\beta \leq 1 \text{ such that } r_{(\alpha,\beta)}A \subseteq \chi_\theta\}$

Hence $\mathrm{ann(A)} = \bigcup\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in [0,1] \text{ with } \alpha+\beta \leq 1 \text{ such that } r_{(\alpha,\beta)}A \subseteq \chi_\theta\}$.

Theorem (3.6) Let M be a R-module and A∈IFS(M). Then $ann(\mathrm{A})\,\mathrm{A} \subseteq \chi_\theta$.

Proof. Now, $(ann(\mathrm{A})A)(x) = (\mu_{ann(\mathrm{A})A}(x), \ \nu_{ann(\mathrm{A})A}(x))$, where

$\mu_{ann(\mathrm{A})A}(x) = \vee\{\mu_{ann(\mathrm{A})}(r) \wedge \mu_A(y) : r \in R, \ y \in M, \ ry = x\}$ and

$\nu_{ann(\mathrm{A})A}(x) = \wedge\{\nu_{ann(\mathrm{A})}(r) \vee \nu_A(y) : r \in R, \ y \in M, \ ry = x\}$.

Therefore, $\mu_{ann(A)A}(x) = \vee\{\mu_{ann(A)}(r) \wedge \mu_A(y):\ r \in R,\ y \in M,\ ry = x\}$

$$= \vee\left[\vee\{\mu_B(r): B \in IFS(R),\ BA \subseteq \chi_\theta\} \wedge \mu_A(y),\ r \in R,\ y \in M,\ ry = x\right]$$
$$= \vee\{\mu_B(r) \wedge \mu_A(y): B \in IFS(R),\ BA \subseteq \chi_\theta,\ r \in R,\ y \in M,\ ry = x\}$$
$$\leq \vee\{\mu_{BA}(ry): B \in IFS(R),\ BA \subseteq \chi_\theta,\ r \in R,\ y \in M,\ ry = x\}$$
$$\leq \vee\{\mu_{\chi_\theta}(x):\ BA \subseteq \chi_\theta\}$$
$$= \mu_{\chi_\theta}(x)$$

i.e., $\mu_{ann(A)A}(x) \leq \mu_{\chi_\theta}(x),\ \forall\, x \in M$.

Similarly, we can show that $\nu_{ann(A)A}(x) \geq \nu_{\chi_\theta}(x),\ \forall\, x \in M$.

Thus $(ann(A)A) \subseteq \chi_\theta$.

Corollary (3.7) If $A \in IFS(M)$ be such that $\mu_A(\theta) = 1$ and $\nu_A(\theta) = 0$, then $ann(A)A = \chi_\theta$.

Proof. By Lemma (3.3) we have $\chi_0 \subseteq ann(A)$

$\Rightarrow \quad \mu_{\chi_0}(0) \leq \mu_{ann(A)}(0)$ and $\nu_{\chi_0}(0) \geq \nu_{ann(A)}(0)$

i.e., $1 \leq \mu_{ann(A)}(0)$ and $0 \geq \nu_{ann(A)}(0)$

$\Rightarrow \quad \mu_{ann(A)}(0) = 1$ and $\nu_{ann(A)}(0) = 0$.

Now, $\mu_{ann(A)A}(\theta) = \vee\{\mu_{ann(A)}(r) \wedge \mu_A(m):\ r \in R,\ m \in M,\ rm = \theta\}$

$$\geq \mu_{ann(A)}(0) \wedge \mu_A(\theta)$$
$$= 1 \wedge 1 = 1$$

i.e., $\mu_{ann(A)A}(\theta) = 1$. Similarly, we can show that $\nu_{ann(A)A}(\theta) = 0$. Therefore, $\chi_\theta \subseteq ann(A)A$. Hence by Theorem (3.6) we get $ann(A)A = \chi_\theta$.

Note (3.8) If $A \in IFM(M)$, then $ann(A)A = \chi_\theta$.

Theorem (3.9) Let M is a R-module and $B \in IFS(R)$, $A \in IFS(M)$ such that $BA \subseteq \chi_\theta$ if and only if $B \subseteq ann(A)$.

Proof. By definition of annihilator $BA \subseteq \chi_\theta \Rightarrow B \subseteq ann(A)$.

Conversely, let $B \subseteq ann(A) \Rightarrow BA \subseteq ann(A)A \subseteq \chi_\theta$.

Corollary (3.10) If in the above theorem (3.8) $\mu_B(0) = 1$, $\nu_B(0) = 0$ and $\mu_A(\theta) = 1$, $\nu_A(\theta) = 0$, then $BA = \chi_\theta$ if and only if $B \subseteq ann(A)$.

Theorem (3.11) Let M is a R-module and $A, B \in IFS(M)$. Then the following conditions are equivalent:

(*i*) $ann(B) = ann(A)$, for all $B \subseteq A$, $B \neq \chi_\theta$.

(*ii*) $CB \subseteq \chi_\theta$ implies $CA \subseteq \chi_\theta$, for all $B \subseteq A$, $B \neq \chi_\theta$, $C \in IFS(R)$.

Proof. For (i) $\Rightarrow$ (ii) Let $CB \subseteq \chi_\theta$. Then by theorem (3.9) we have

$C \subseteq ann(B) = ann(A)$ (by (i)). Again by the same theorem we have $CA \subseteq \chi_\theta$.

For (ii) $\Rightarrow$ (i) By theorem (3.6) we have $ann(B)B \subseteq \chi_\theta$.

So (ii) implies $ann(B)A \subseteq \chi_\theta$ where $B \subseteq A$, $B \neq \chi_\theta$.

By theorem (3.8) $ann(B) \subseteq ann(A)$.

Also, $B \subseteq A \Rightarrow ann(A) \subseteq ann(B)$. Thus $ann(A) = ann(B)$.

Corollary (3.12) If in the above theorem $\mu_A(\theta) = 1$, $\nu_A(\theta) = 0$ and $\mu_B(\theta) = 1$, $\nu_B(\theta) = 0$. Then the above theorem can be stated as: The following conditions are equivalents:

(i) $ann(B) = ann(A)$, for all $B \subseteq A$, $B \neq \chi_\theta$.

(ii) $CB = \chi_\theta$ implies $CA = \chi_\theta$, for all $B \subseteq A$, $B \neq \chi_\theta$, $C \in IFS(R)$ with $\mu_C(0) = 1$, $\nu_C(0) = 0$.

Theorem (3.13) Let M is a R-module and $A \in IFS(M)$. Then

$ann(A) = \bigcup\{B: B \in IFI(R) \text{ such that } BA \subseteq \chi_\theta\}$, where IFI(R) is the set of intuitionistic fuzzy ideals of R.

Proof. Clearly, $\bigcup\{B: B \in IFI(R) \text{ such that } BA \subseteq \chi_\theta\} \subseteq \bigcup\{B: B \in IFS(R) \text{ such that } BA \subseteq \chi_\theta\} = ann(A)$.

Let $r \in R, \alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ such that $r_{(\alpha,\beta)}A \subseteq \chi_\theta$.

Let $B = < r_{(\alpha,\beta)} >$. Then $< r_{(\alpha,\beta)} > A = (\alpha,\beta)_{<r>}A$.

$$\text{Again, } \mu_{(\alpha,\beta)_{<r>}A}(x) = \vee\{\mu_{(\alpha,\beta)_{<r>}}(s) \wedge \mu_A(y) \mid r \in R,\ y \in M,\ sy = x\}$$
$$= \vee\{\alpha \wedge \mu_A(y) \mid s \in < r >,\ y \in M,\ sy = x\}$$
$$\leq \vee\{\mu_{r_{(\alpha,\beta)}A}(ry) \mid t \in R,\ y \in M,\ \mathrm{t}(ry) = x\}$$
$$\leq \vee\{\mu_{\chi_\theta}(ry) \mid t \in R,\ y \in M,\ \mathrm{t}(ry) = x\}$$
$$\leq \vee\{\mu_{\chi_\theta}(\mathrm{t}(ry)) \mid t \in R,\ y \in M,\ \mathrm{t}(ry) = x\}$$
$$= \mu_{\chi_\theta}(x).$$

Thus, $\mu_{(\alpha,\beta)_{<r>}A}(x) \leq \mu_{\chi_\theta}(x)$. Similarly, we can show that $\nu_{(\alpha,\beta)_{<r>}A}(x) \geq \nu_{\chi_\theta}(x), \forall x \in M$.

Therefore, we have $(\alpha,\beta)_{<r>}A \subseteq \chi_\theta$.

Hence $\bigcup\{B: B \in IFI(R) \text{ such that } BA \subseteq \chi_\theta\} \supseteq$

$\bigcup\{r_{(\alpha,\beta)}: r \in R\ \alpha,\beta \in [0,1] \text{ with } \alpha + \beta \leq 1 \text{ such that } r_{(\alpha,\beta)}A \subseteq \chi_\theta\} = ann(A)$.

Hence $ann(A) = \bigcup\{B: B \in IFI(R) \text{ such that } BA \subseteq \chi_\theta\}$.

Theorem (3.14) Let M is a R-module and $A \in IFS(M)$. Then $ann(A) \in IFI(R)$.

Proof. Since $\chi_0 A \subseteq \chi_\theta$, so $\chi_0 \subseteq ann(A)$. Let $r_1, r_2 \in R$ be any elements. Then,

$\mu_{ann(A)}(r_1)\wedge\mu_{ann(A)}(r_2)$

$=\left(\vee\left\{\mu_{A_1}(r_1): A_1\in IFI(R), A_1A\subseteq\chi_\theta\right\}\right)\wedge\left(\vee\left\{\mu_{A_2}(r_2): A_2\in IFI(R), A_2A\subseteq\chi_\theta\right\}\right)$

$=\vee\left\{\mu_{A_1}(r_1)\wedge\mu_{A_2}(r_2): A_1, A_2\in IFI(R), A_1A\subseteq\chi_\theta, A_2A\subseteq\chi_\theta\right\}$

$\le\vee\left\{\mu_{A_1+A_2}(r_1)\wedge\mu_{A_1+A_2}(r_2): A_1, A_2\in IFI(R), A_1A\subseteq\chi_\theta, A_2A\subseteq\chi_\theta\right\}$

$\le\vee\left\{\mu_{A_1+A_2}(r_1-r_2): A_1+A_2\in IFI(R), A_1A+A_2A\subseteq(A_1+A_2)A\subseteq\chi_\theta+\chi_\theta=\chi_\theta\right\}$

$\le\vee\left\{\mu_B(r_1-r_2): B\in IFI(R), BA\subseteq\chi_\theta\right\}$

$=\mu_{ann(A)}(r_1-r_2).$

Thus, $\mu_{ann(A)}(r_1-r_2)\ge\mu_{ann(A)}(r_1)\wedge\mu_{ann(A)}(r_2).$

Similarly, we can show that $\nu_{ann(A)}(r_1-r_2)\le\nu_{ann(A)}(r_1)\wedge\nu_{ann(A)}(r_2).$

Again, $\mu_{ann(A)}(sr)=\vee\{\mu_B(sr): B\in IFI(R), BA\subseteq\chi_\theta\}\ge\vee\{\mu_B(r): B\in IFI(R), BA\subseteq\chi_\theta\}=\mu_{ann(A)}(r).$

Thus, $\mu_{ann(A)}(sr)\ge\mu_{ann(A)}(r)$. *Similarly*, we can show that $\nu_{ann(A)}(sr)\le\nu_{ann(A)}(r),\ \forall\, r,s\in R.$

Hence $ann(A)\in IFI(R).$

Theorem (3.15) Let M is a R-module and $A_i\in$ IFS(M), i $\in\Lambda$. Then

$$ann\left(\bigcup_{i\in\Lambda}A_i\right)=\bigcap_{i\in\Lambda}ann(A_i).$$

Proof. $ann\left(\bigcup_{i\in\Lambda}A_i\right)=\bigcup\left\{\text{B}:\ \text{B}\in IFS(R)\text{ such that }B\left(\bigcup_{i\in\Lambda}A_i\right)\subseteq\chi_\theta\right\}$

$=\bigcup\left\{\text{B}:\ \text{B}\in IFS(R)\text{ such that }\bigcup_{i\in\Lambda}BA_i\subseteq\chi_\theta\right\}$

$\subseteq\bigcup\left\{\text{B}:\ \text{B}\in IFS(R)\text{ such that }BA_i\subseteq\chi_\theta\right\}$

$=ann(A_i),\ \ \forall\ i\in\Lambda.$

Hence $ann\left(\bigcup_{i\in\Lambda}A_i\right)\subseteq\bigcap_{i\in\Lambda}ann(A_i).$

By Theorem (3.6), we have

$$\left(\bigcap_{i\in\Lambda}ann(A_i)\right)\left(\bigcup_{j\in\Lambda}A_j\right)=\bigcup_{j\in\Lambda}\left(\bigcap_{i\in\Lambda}ann(A_i)A_j\right)\subseteq\bigcup_{j\in\Lambda}\left(ann(A_j)A_j\right)\subseteq\bigcup_{j\in\Lambda}\chi_\theta=\chi_\theta.$$

Thus $\bigcap_{i\in\Lambda}ann(A_i)\subseteq ann\left(\bigcup_{i\in\Lambda}A_i\right).$

Hence $ann\left(\bigcup_{i\in\Lambda}A_i\right)=\bigcap_{i\in\Lambda}ann(A_i).$

Theorem (3.16) Let M is a R-module and A, B $\in$ IFM(M), then

ann(A+B) = ann(A) $\cap$ ann(B).

Proof. Since A, B $\in$ IFM(M) $\Rightarrow$ A + B $\in$ IFM(M), we have

$\mu_{A+B}(x) = \vee_{x=y+z}\{\mu_A(y) \wedge \mu_B(z)\} \geq \mu_A(x) \wedge \mu_B(\theta) = \mu_A(x)$ and

$\nu_{A+B}(x) = \wedge_{x=y+z}\{\nu_A(y) \vee \nu_B(z)\} \leq \nu_A(x) \vee \nu_B(\theta) = \nu_A(x), \ \forall \ x \in M$.

This implies that $A \subseteq A+B$ and $B \subseteq A+B$.

So, $\text{ann}(A+B) \subseteq \text{ann}(A)$ and $\text{ann}(A+B) \subseteq \text{ann}(B)$

$\Rightarrow \ \text{ann}(A+B) \subseteq \text{ann}(A) \cap \text{ann}(B)$.

Now, $\text{ann}(A) \cap \text{ann}(B)$

$= \left(\cup\{A_1 \mid A_1 \in IFI(R), A_1A \subseteq \chi_\theta\}\right) \cap \left(\cup\{B_1 \mid B_1 \in IFI(R), B_1B \subseteq \chi_\theta\}\right)$

$= \cup\{A_1 \cap B_1 \mid A_1, B_1 \in IFI(R), A_1A \subseteq \chi_\theta, B_1B \subseteq \chi_\theta\}$

$\subseteq \cup\{C \mid C = A_1 \cap B_1 \in IFI(R), CA \subseteq \chi_\theta, CB \subseteq \chi_\theta\}$

$\subseteq \cup\{C \mid C = A_1 \cap B_1 \in IFI(R), C(A+B) \subseteq CA + CB \subseteq \chi_\theta\}$

$= \cup\{C \mid C \in IFI(R), C(A+B) \subseteq \chi_\theta\}$

$= ann(A+B)$.

Therefore, $\text{ann}(A) \cap \text{ann}(B) \subseteq ann(A+B)$.

Hence $ann(A+B) = \text{ann}(A) \cap \text{ann}(B)$.

Definition (3.17) Let M be R-module. Then A$\in$ IFS(M) is said to be faithful if ann(A) = χ_0.

Lemma (3.18) Let A$\in$ IFS(M) be faithful, where M is R-module. If R is non-zero then A $\neq \chi_\theta$.

Proof. Since A is faithful $\Rightarrow$ ann(A) = χ_0.

If A = χ_θ then ann(A) = ann(χ_θ) = χ_R. Thus we have $\chi_0 = \chi_R \Rightarrow$ R = {0}, a contradiction.

Therefore, A $\neq \chi_\theta$.

Theorem (3.19) Let A $\in$ IFS(R) with $\mu_A(0) = 1$, $\nu_A(0) = 0$. Then A $\subseteq$ ann(ann(A)) and ann (ann(ann(A))) = ann(A).

Proof. Let A be an intuitionistic fuzzy subset of R-module R. Then by corollary (3.7), we have ann(A)A = χ_0.

By theorem (3.9), we have A $\subseteq$ ann(ann(A)) ....(1)

$\Rightarrow$ ann(ann(ann(A))) $\subseteq$ ann(A) [using lemma (3.4)]

Again using (1) : ann(A) $\subseteq$ ann(ann(ann(A))).

So, ann (ann(ann(A))) = ann(A).

Theorem(3.20) Let A $\in$ IFS(M). Then

$C_{(\alpha,\beta)}(ann(A)) \subseteq ann(C_{(\alpha,\beta)}(A)), \forall \ \alpha, \beta \in (0,1]$ with $\alpha + \beta \leq 1$.

Proof. Let $x \in C_{(\alpha,\beta)}(ann(A))$. *Then* $\mu_{ann(A)}(x) \geq \alpha > 0$ and $\nu_{ann(A)}(x) \leq \beta < 1$

$\Rightarrow \vee\{\mu_B(x) : B \in IFI(R), BA \subseteq \chi_\theta\} \geq \alpha$ and $\wedge\{\nu_B(x) : B \in IFI(R), BA \subseteq \chi_\theta\} \leq \beta$

$\Rightarrow \mu_B(x) \geq \alpha$ and $\nu_B(x) \leq \beta$ for some $B \in IFI(R)$ with $BA \subseteq \chi_\theta$.

If $x \notin ann(C_{(\alpha,\beta)}(A))$ then $\exists$'s some $y \in C_{(\alpha,\beta)}(A)$ such that $xy \neq \theta$.

Now, $\mu_{BA}(xy) \geq \mu_B(x) \wedge \mu_A(y) \geq \alpha > 0$ and $\nu_{BA}(xy) \leq \nu_B(x) \vee \nu_A(y) \leq \beta < 1$,

which is a contradiction. Hence $C_{(\alpha,\beta)}(ann(A)) \subseteq ann(C_{(\alpha,\beta)}(A))$.

Definition (3.21) A$\in$IFI(R) is said to be an intuitionistic fuzzy dense ideal if ann(A) = χ_0 .

Definition (3.22) A$\in$IFI(R) is called intuitionistic fuzzy semiprime ideal of R if for any IFI B of R such that $B^2 \subseteq$ A implies that B $\subseteq$ A.

Theorem (3.23) If A is an IFI of a semi prime ring R, then A $\cap$ ann(A) = χ_0 and A + ann(A) is an intuitionistic fuzzy dense ideal of R.

Proof. Since A $\cap$ ann(A) $\subseteq$ A , A $\cap$ ann(A) $\subseteq$ ann(A) so $(A \cap ann(A))^2 \subseteq$ A ann(A) $\subseteq \chi_0$.

Now R is a semiprime ring and it implies that 0 is a semiprime ideal of R so χ_0 is an intuitionistic fuzzy semi prime ideal of R.

Also, $(A \cap ann(A))^2 \subseteq \chi_0 \Rightarrow$ A $\cap$ ann(A) $\subseteq \chi_0$ and hence A $\cap$ ann(A) = χ_0 .

Hence ann(A + ann(A)) = ann(A) $\cap$ ann(ann(A)) = χ_0 proving thereby A+ ann(A) is an intuitionistic fuzzy dense ideal of R.

Theorem (3.24) Let A be a non-zero intuitionistic fuzzy ideal of a prime ring R with $\mu_A(0) = 1$, $\nu_A(0) = 0$. Then A is an intuitionistic fuzzy dense ideal of R.

Proof. Now, A ann(A) = χ_0 $\Rightarrow$ ann(A) = χ_0 or A = χ_0. But A $\neq$ χ_0 so ann(A) = χ_0.

Hence A is an intuitionistic fuzzy dense ideal of R.

Definition (3.25) If A$\in$ IFS(R). Then the intuitionistic fuzzy ideal of the form ann(A) is called an intuitionistic fuzzy ideal. Thus if A is an intuitionistic fuzzy annihilator ideal if and only if A = ann(B) for some B $\in$IFS(R) with with $\mu_B(0) = 1$, $\nu_B(0) = 0$.

Remark (3.26) In view of theorem (3.19) it follows that A is an annihilator ideal of R implies ann(ann(A)) = A.

Theorem (3.27) The annihilator ideals in a semiprime ring form a complete Boolean algebra with intersection as infimum and ann as complementation.

Proof. Since $\bigcap_{i \in I} ann(A_i) = ann\left(\sum_{i \in I} A_i\right)$, so any intersection of annihilator ideals is an intuitionistic fuzzy annihilator ideal. Hence these ideals form a complete semi-lattice with intersection as infimum. To show that they form a Boolean algebra it remain to show that:

$A \cap$ ann(B) = χ_0 if and only if $A \subseteq B$, for any annihilator ideals A and B.

If $A \subseteq B$ then $A \cap$ ann(B) $\subseteq B \cap$ ann(B) = χ_0.

Conversary, let $A \cap$ ann(B) = χ_0.

Now, A ann(B) $\subseteq$ $A \cap$ ann(B) = χ_0 $\Rightarrow$ $A \subseteq$ ann(ann(B)) = B.

Theorem (3.28) Let M be a non-zero R-module. Suppose that there exist no ideal A maximal among the annihilators of non-zero intuitionistic fuzzy submodules (IFSMs) of M. Then A is an intuitionistic fuzzy prime ideal of R.

Proof. Since A is maximal among the annihilators of non-zero intuitionistic fuzzy submodules (IFSMs) of M. Therefore there is an IFSM B ($\neq \chi_0$) of M such that A = ann(B).

Suppose P, Q $\in$ IFI(R) properly containing A (i.e., $A \subset P$ and $A \subset Q$) such that $PQ \subseteq A$.

If QB = χ_0 then $Q \subseteq$ ann(B) = A, which is a contradiction to our supposition so QB $\neq \chi_0$.

Now, $PQ \subseteq A$ $\Rightarrow$ P(QB) $\subseteq$ AB = ann(B)B = QB = χ_0. So $Q \subseteq$ ann(QB).

Hence $A \subseteq$ ann(QB). This is a contradiction of the maximality of A. So A is an intuitionistic fuzzy prime ideal of R.

Remark (3.29) If $A \in$ IFM(M) , $A \neq \chi_\theta$ satisfying one (hence both) the condition of Theorem (3.11) then A is called an intuitionistic fuzzy prime submodule of M.

Theorem (3.30) If A is an intuitionistic fuzzy prime submodule of M then ann(A) is an intuitionistic fuzzy prime ideal of R.

Proof. Let A be an intuitionistic fuzzy prime submodule of M and PQ $\subseteq$ ann(A) , where Q is not contained in ann(A). Then

$\chi_\theta \neq QA \subseteq A$. Now PQ $\subseteq$ ann(A) $\Rightarrow$ (PQ)A $\subseteq$ ann(A)A = χ_θ .

So, P $\subseteq$ ann(QA) = ann(A), as A is prime. Hence ann(A) is prime ideal of R.

4. CONCLUSIONS

In this paper we have developed the notion of annihilator of an intuitionistic fuzzy subset of a R-module. Using this notion, we investigate some important characterization of intuitionistic fuzzy annihilator of subsets of modules. The annihilator of union (sum) of intuitionistic fuzzy submodules are obtained. Annihilator of intuitionistic fuzzy ideal of prime ring, semi prime ring are also obtained. Using the concept of intuitionistic fuzzy annihilators, intuitionistic fuzzy prime submodules and intuitionistic fuzzy ideals are defined and various related properties are established.

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